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Acta Psychologica 110 (2002) 35–61

acta psychologica

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# The perception of closed flat knots and completion by folding

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Received 7 November 2000; received in revised form 20 August 2001; accepted 21 August 2001

**Abstract**

The aim of the present research was to study a well-defined set of line drawings that have never been analyzed before, and that are perceived as closed flat knots (CFKs). These knots are obtained by folding two-dimensional laminae. From this study it emerged that the perception of CFKs is always accompanied by a type of amodal completion, which had not previously been studied, and which we have called "completion by folding". It occurs when three quadrilaterals are so arranged that they look like a sheet of paper folded around one of them, partly occluded and partly occluding. Two experiments were carried out on this phenomenon revealing that a three-level stratification is produced in the completion by folding that facilitates unification between occluded figures, even when current models do not foresee such a perceptual solution, either because the distance between the two figures that become a unit is too big (experiment 1) or because the slope between the two figures is not favorable (experiment 2). A third experiment, which took up the problem of the perception of CFKs, revealed the following: (1) Besides the prototype CFK, obtained by the interlacing of a rectangular lamina, there is a theoretically infinite class of CFKs. (2) The drawing of all the possible closed flat knots involves geometrically precise rules. (3) Not all feasible knots with these rules are seen as such. (4) Only the knots with characteristics of alignment and regularity are recognized as knots. (5) The closed flat knots are seen as the result of a transformation which the figure undergoes. This evidence constitutes a remarkable perceptual problem, discussed with reference to the recent theories of amodal completion. 

*PsycINFO classification:* 2323

*Keywords:* Visual perception; Amodal completion; Completion by folding

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0001-6918/02/$ - see front matter 
PII: S0001-6918(01)00069-5

## 1. Introduction

### *1.1. From perceptual completion to knots*

The sensorial registration of objects that we observe everyday is most often incomplete, yet we almost never have the impression of seeing only some parts of these objects. A chair partially hidden by a table or a car parked behind a tree does not appear as if they were missing parts that we do not see. In the same way, when we look at the facade of a church, this appears complete even if we cannot see the apse on the opposite side. The process by which we perceive partially hidden as complete was called "amodal completion" by Michotte, Thinès, and Crabbé (1964).

The phenomenon is so common and familiar that, as Gibson (1966, p. 203) says, it does not appear as a problem for those who are not interested in perception. Sekuler and Palmer (1992), who are instead interested in this topic, assert that phenomenological demonstrations are unfortunately inadequate to explain completion in terms of underlying physiological processes: "phenomenology of perceptual completion is so compelling that many psychologists seem to be satisfied with mere existence proofs, demonstration based purely on appeals to subjective evidence" (p. 95). According to these authors, the questions that one should start to answer are of this kind: "In which objective sense does completion occur?" "How does this process unfold in time?" "What mechanisms does the process consist of?" (p. 95). One cannot but agree with the spirit of such an invitation, but one must also be careful that the urge to explain things, before having spotted all the phenomena, will not lead to "an excessive production of models, whose life is extremely brief, which corresponds to a too limited collection of new data and new phenomena" (Uttal, 1988).

In the following pages two phenomena that belong to the domain of amodal completion will be discussed, even if up to now it would seem that the most accredited of theories cannot explain them.

There are two kinds of completion, because there are two kinds of occlusion: (a) self-occlusion, which always happens when there is a solid non-transparent object which occludes part of itself to an observer; (b) hetero-occlusion, produced by an object situated between an observer and another more distant object, so that part of the latter is hidden. Phenomenal folding is a particular case of completion, half way between self-occlusion and hetero-occlusion.

Phenomenal foldings are line drawings showing folded laminae or sheets, representing a peculiar case of interposition between two figures which are not reciprocally independent, but are connected to each other. The first studies of this phenomenon (Massironi, 1988; Massironi & Bruno, 1997) defined the conditions that a pattern must have in order to show a clear case of phenomenal folding. These conditions are summarized in Appendix A.

Phenomenal knots are special cases of folding. See Fig. 1(a)–(b), in which the same pattern is drawn on a white background. When asked what these images represent, about 75% of observers recognize the image as a knot. The remaining 25% do not see any unequivocally defined figure; however, as soon as the suggestion of a

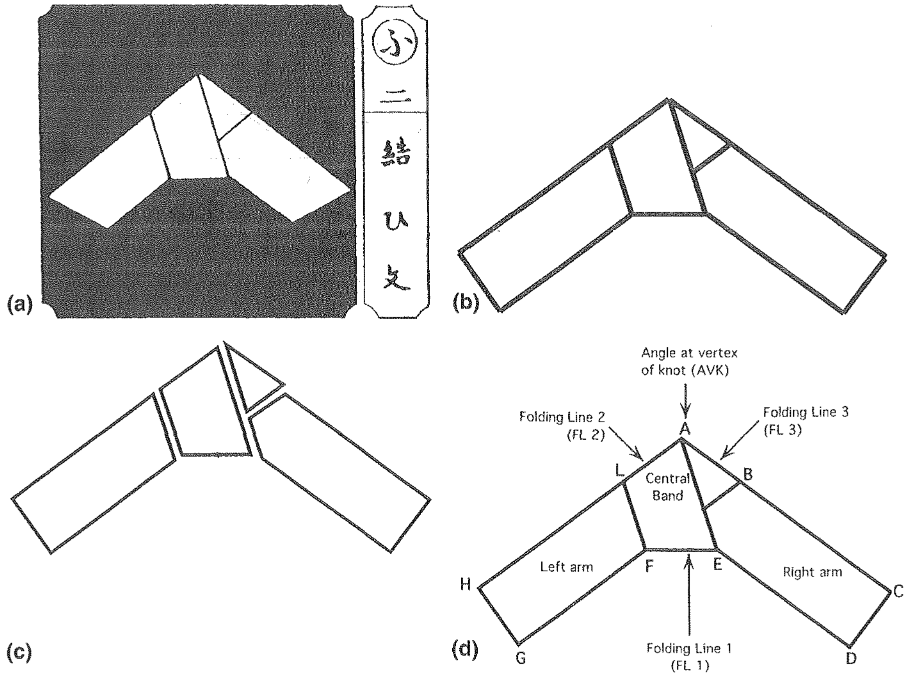


Fig. 1. A traditional Japanese emblem showing a knot (a); a flat closed knot (b); the four parts that make up a phenomenal knot presented as a mosaic (c); terms used to define the structural parts of knots (d).

knot is made, they are able to see it clearly. A knot does not consist of a single folding, but of three foldings and an interlacement.

The figure as a whole does not seem particularly "good" from a Gestalt point of view, since it consists of four shapes that are not perfectly regular and are all different from each other. They should favor the percept of a mosaic, as in Fig. 1(c), but this does not happen. The figure is seen as a unitary structure in which the four parts are strictly connected in the same figural unity. This substantial perceptual result involves the need to understand what information the system uses to come to this specific perceptual result.

In order to make the writer's, and especially the reader's, task easier, it would be opportune to introduce two handy tools. The first consists of an essential glossary of the terms that will be used to refer to the parts of a knot [see Fig. 1(d)]. The second tool is the definition of the set of knots under examination, which will be defined as "Closed Flat Knots" (from now on CFKs). Flat, because these knots are obtainable from two-dimensional strips like sheets of paper, which can be modeled, folded and interlaced in such a way that, once they are closed, they present no curlings or crumplings in the third dimension, apart from simple overlapping. Knots obtained from ropes, ties and strips of thicker material cannot, therefore, be considered as flat knots

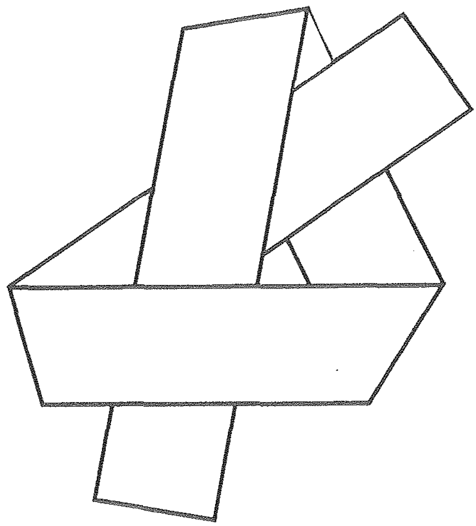

Fig. 2. A flat non-closed knot.

(Adams, 1994). Closed, because these knots are those in which the ends of each FL touch the contour of the folded band. Fig. 2 shows an example of a "non-closed knot".

So far Fig. 1(a) and (b) is the only example of a closed flat knot.

In order to understand which information the visual system uses to perceive a knot, we have taken it apart to discover its characteristic geometric constants.

### *1.2. The alignments*

The structural appearance which is evident in the shape of the knot is produced by the alignment and recurrent connections between the vertexes of the polygons that constitute it. By assigning a letter to each vertex [as in Fig. 1(d)] it may be seen that these are aligned three at a time, that is: A–B–C, D–E–L, B–F–G, A–L–H. Moreover, points A, E are connected with points E, F, respectively.

### *1.3. Completion by folding*

Fig. 1 reveals that the most informative parts of the pattern are the three folding lines. Fig. 3 shows what takes place in relation to the three folding lines.

It is a particular case of amodal completion which will be called "completion by folding".

To explain this new phenomenon, the starting point must be typical examples of amodal completion – with or without unification – as seen in Fig. 4(a) and (b), in which the margins of the phenomenally occluded part meet an externally occluding

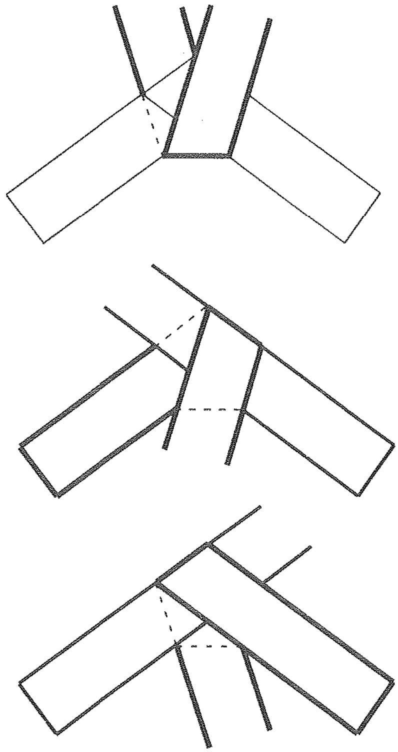

Fig. 3. Amodal completion by folding in correspondence to the three folding lines of a knot.

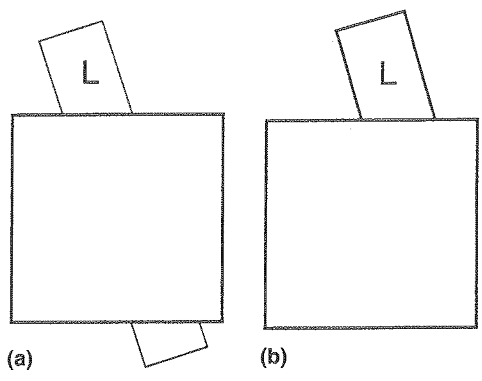


Fig. 4. An example of classic amodal completion (a)–(b).

side of an overlapping square. In Fig. 5, the upper side of the trapezium "L" meets a side of a square from the inside. From a perceptual point of view, the segment which

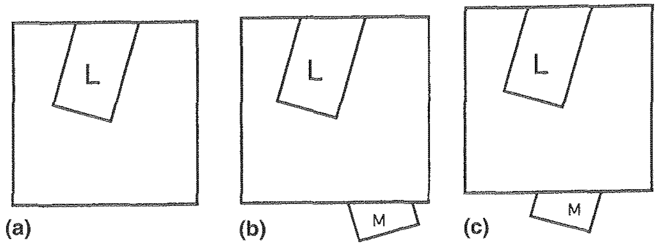


Fig. 5. An ambiguous case of overlapping (b); examples of amodal completion by folding (b)–(c) (see explanation in the text).

is common to both figures assumes an uncertain position. In fact, it is not sure whether it is seen as a side of "L", that coincides with the upper side of the square, or whether it is seen as a folding line of "L" that, when folded, is amodally completed behind the square.

In the former case there is some resistance in agreement with both Rock's rejection of the coincidence principle (Rock, 1983, p. 137) and what is foreseen by the theory of non-accidental properties (Lowe, 1987; Wagemans, 1992, 1993; Witkin & Tenenbaum, 1983).

In the case of Fig. 5(a), in which both rejection of coincidences and non-accidental conditions are present, there is a lack, however, of a visual anchor that would stabilize the perception of the common segment as a folding margin. The latter solution, instead, becomes immediately coercive if a rectangular or trapezoidal shape "M" is added to the side of the square opposite the one on which "L" lies [Fig. 5(b)].

In this case the quadrilateral "M" is seen to complete itself with "L" under the square, even if the correspondences and good continuation between the ends of "M" and those of "L" are not respected [Fig. 5(c)]. This is a "completion by folding" in which the occluding figure is seen, in its turn, to be occluded by the figure which it occludes.

This new type of completion requires further analysis.

## 2. Experiment 1

Preliminary observations showed that, given two independent quadrilaterals with the upper sides lying one upon the prolongation of the other, by adding a rectangle between the two margins, it appeared as if they were part of a single lamina or sheet folded over the horizontal rectangle. It was also observed that the unification occurred, even when the distance between the two quadrilaterals became considerably larger.

The purpose of this experiment was to test the role played by an interposed figure in the phenomenal unification of two quadrilaterals on a single lamina folded over the first.

*2.1. Methods*

*2.1.1. Subjects*

Twenty students from the University of Verona, aged between 18 and 33 (10 females and 10 males), took part in the experiment.

*2.1.2. Material*

Twelve line drawings depicting the patterns represented in Fig. 6 were drawn with black ink on white cardboard cards that measured 13 × 13 $cm^2$. The patterns were obtained by systematically modifying the phenomenal folding depicted in pattern "a" in Fig. 6.

The modifications of such a pattern involved two variables:

(A) The degree of coincidence of the upper horizontal side of the two quadrilaterals with four levels, i.e.: (1) complete coincidence of the two sides (patterns a, e, i); (2) partial coincidence of the two sides (patterns b, f, l); (3) only one point of coincidence of the two sides (patterns c, g, m); (4) no coincidence between the two sides (patterns d, h, n).

(B) Characteristics of the interposed figure with three levels: (1) no figure interposed (patterns a, b, c, d); (2) a segment (patterns e, f, g, h); (3) a rectangle (patterns i, l, m, n).

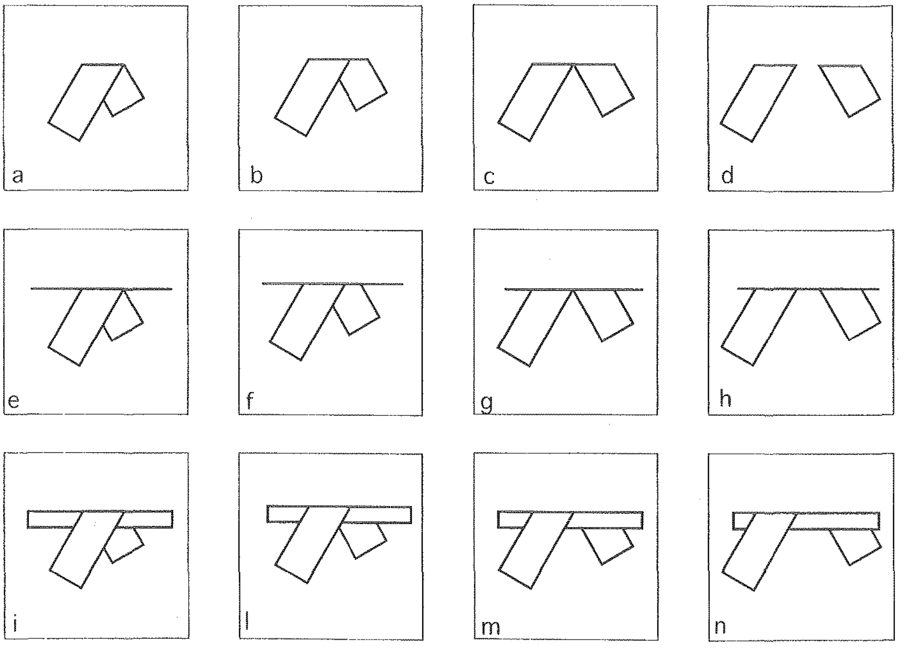

Fig. 6. Stimuli used in the first experiment.

The combination of the four levels of the first variable with the three of the second produced the 12 stimuli of Fig. 6. Besides this experimental material, other six cardboard cards were drawn for training purposes. These showed: (i) two flat juxtaposed figures; (ii) two flat figures with a point in common; (iii) two flat figures, one of which partially superimposed on the other; (iv) two flat figures tangent to a segment; (v) a case of phenomenal folding; (vi) a case of completion by folding, different from the ones used as experimental material.

### *2.1.3. Procedure*

The training phase was followed by three experimental sessions. The training phase and the first experimental session consisted of the attribution of a verbal description, expressly arranged beforehand, to each of the stimuli presented in sequence. A piece of paper was given to the subjects at the beginning of the trials on which four different descriptions were written in random order for each subject. The descriptions were:

(D1) A flat form folded on itself.
(D2) A flat form folded over a line or a rectangle.
(D3) Two flat forms which were either independent or juxtaposed, or tangent, or partially superimposed.
(D4) Two independent flat forms, tangent to a line or to a rectangle.

The subjects' task was to decide which of the descriptions best fitted the configurations presented to them in random order.

For the second and third experimental sessions, the subjects were told that they would see the same stimuli they had seen during the first session, but this time they would be presented three at a time for the second session, and four at a time for the third. The subjects' task was to order the triads – and the quadruplets – of stimuli, according to whether they thought the two critical quadrilaterals belonged to the same physical structure or not. In order to define this aspect, the term of mutual "belongingness" was adopted. In this way, the fact that the two quadrilaterals are seen to belong to the same phenomenal object was highlighted. The term "completion", instead, would seem to be a better definition of cases in which a single object emerges from the occluding figure. In fact, the subjects understood their task better when dealing with "belongingness" as compared to "completion". The former is more appropriate to define a scalar quantity, whereas the latter would seem better to describe all or no conditions.

Subjects were asked to start their order with the stimulus that most looked as if it were depicting a single lamina. In the groups of three stimuli per session, they corresponded to each column in Fig. 6, while the groups of four stimuli per session corresponded to the rows in the same figure. Once subjects had ordered a triad or quadruplet, they were asked to evaluate the intensity of the "belongingness" effect for each stimulus, by means of a 10 point scale, in which "10" meant that the two quadrilaterals were definitely part of the same lamina, while "1" meant that the

Table 1
The frequencies with which the descriptions were associated to the different stimuli

| Description | a | b | c | d | e | f | g | h | i | l | m | n |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| D1 | 19 | 10 | 1 | | 2 | | | | | 2 | | |
| D2 | 1 | 3 | | | 18 | 15 | 3 | | 20 | 18 | 16 | 10 |
| D3 | | 4 | 7 | 20 | | 1 | | | | | 3 | 7 |
| D4 | | 3 | 12 | | | 4 | 17 | 20 | | | 1 | 3 |
| | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 |

two quadrilaterals were completely independent of each other, that is they were separate objects. All subjects performed the three tasks with no difficulty.

### 2.2. *Results*

The frequencies with which the descriptions were associated to the different stimuli are shown in Table 1. In Table 1, the column headings report the labels for the stimuli as seen in Fig. 6, while the row headings indicate the four descriptions as reported in the procedure paragraph.

In the cells reporting the frequencies equal to or over 15 (shaded cells in Table 1), description D2 which refers to "flat folded forms......" was associated not only with the stimuli "e" and "f", but also with the stimuli "i", "l", "m" of Fig. 6. The former involve figures that are seen to be folded even without interposing elements, whereas the latter three are configurations that are not seen as folds without the interposing figures.

A few analyses of the clusters were calculated on the frequencies in Table 1 which were transformed into logarithmic values. Since there are only three degrees of freedom in these findings, they have been reduced to three dependent variables, according to the following formula: $e1 = \ln(\mathrm{D1} + 1/6)/(\mathrm{D4} + 1/6)$; $e2 = \ln(\mathrm{D2} + 1/6)/(\mathrm{D4} + 1/6)$; $e3 = \ln(\mathrm{D3} + 1/6)/(\mathrm{D4} + 1/6)$ (Agresti, 1996).

In Fig. 7 is shown the dendrogram of the hierarchic cluster analysis calculated on the square of the Euclid distances, using the average linkage method. This analysis was the one which best grouped the frequencies, according to which descriptions were associated to the stimuli.

Three clusters were identified: the first may be called perfect folding, in that it groups the stimuli e, l, i, a, which, with the exception of l, are all cases of classic folding. The second, or approximate folding, groups the stimuli m, n, f, b. The third, which may be called partially independent forms, only includes the stimuli g and h. The stimuli c and d remain ungrouped at this level.

The ranking of the groups of four stimuli (rows) was in total accordance with the order given in Fig. 6.

Kendall's '$W$' coefficient (Siegel & Castellan, 1988) was used to test the agreement between the ranking performed by all subjects. The agreement between the rankings for rows 1 and 3 of Fig. 6 was total ($W = 1$) and almost total for row 2 [$W = 0.93$;

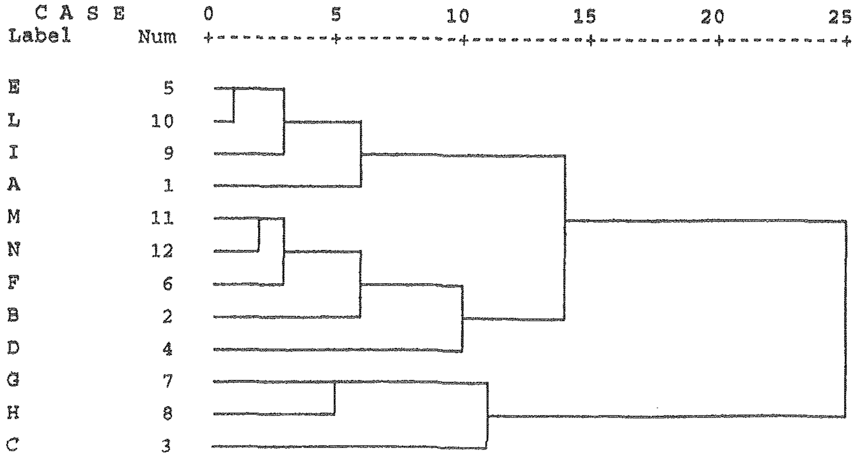


Fig. 7. Dendrogram of the cluster analysis of the results obtained in experiment 1.

Chi-squared (DF, 6) = 112.02, $p < 0.001$]. This means that the distance factor always influenced the subjects' rankings in the same way.

The same kind of analyses was performed on the rankings for the triplets of stimuli, corresponding to the patterns considered in each column of Fig. 6. The results are reported in Table 2, which shows that the agreement among subjects was very high and statistically significant.

### *2.3. Discussion*

This first experiment revealed that the stimuli i–n of Fig. 6 simultaneously give rise to two amodal completions: (a) one from the two parts of the horizontal band behind the bigger trapezium, (b) one from the two critical figures hanging from the edge of the upper rectangle. In this way a considerable simplification of the whole pattern was achieved helped by both local aspects like the 'T-junction' between the critical quadrilaterals and the interposed figure (Kanizsa, 1979), and overall aspects like the total reduction of the phenomenal units in the configuration.

Since the role of the interposing element varied according to whether there was a segment or a rectangle, it had to be verified whether the effect was influenced by the width of the interposing figure. The shape in the foreground appears to continue

Table 2
The agreement level (Kendall's $W$ agreement test) reached by subjects in ordering the three patterns having the same degree of coincidence

| Stimuli | $W$ | $X^2$ (DF = 3) | Significant difference |
|---|---|---|---|
| i, e, a | 0.39 | 25.23 | $p < 0.001$ |
| l, f, b | 0.71 | 43.47 | $p < 0.001$ |
| m, g, c | 0.64 | 39.48 | $p < 0.001$ |
| n, h, d | 1 | | $p < 0.001$ |

beneath itself and the interposing figure to join up with the part which is seen to lie on a third plane. This is the result of a specific organizational process, which tends towards simplification, solving, in three-dimensional space, the geometric contradiction of the lack of unification between the prolongations of the two quadrilaterals (see patterns m and n of Fig. 6). It is no easy matter to understand why this mechanism works so efficiently. The system is not very precise in its creation of this type of unification, but it is probably more useful to have approximate simplifications rather than having to respect more rigorous geometric conditions. It is a question of "approximate solutions", not at all exceptional for our perceptual system, which often seems to work like a "sloppy geometer" (Perkins, 1972; Perkins & Cooper, 1980), or which appears to be satisfied with using a "bag of tricks" (Ramachandran, 1985) or even which leads to perceptive heuristics that are "not entirely explicable" (Proffitt, Rock, Hecht, & Schubert, 1992). According to Wagemans and Kolinsky (1994), it is a system that can use different kinds of information and activate different elaborations on the bases of the task that it has to fulfill.

## 3. Experiment 2

The second experiment, besides verifying the effect of the width of the interposing element, aimed at finding out whether the perceptual result was influenced by the reciprocal inclination of the two critical quadrilaterals.

### 3.1. Methods

#### 3.1.1. Subjects

Sixteen students from the University of Verona took part in the experiment (8 males and 8 females).

#### 3.1.2. Material

As in the previous experiment, the stimuli were patterns drawn with black ink on a white cardboard card ($21 \times 21$ cm$^2$). There were 16 stimuli, all of which were modifications of a standard pattern depicting a rectangular strip, that measured $9 \times 2$ cm$^2$, folded over a horizontal folding line. The angle between the two interlaced quadrilaterals measured 56° (see Fig. 8, stimulus B1, shaded cell).

The stimuli in the first row of Fig. 8 were derived from the standard (B1) by holding the left quadrilateral constant and rotating the right one 15° counterclockwise, around the central point of the upper folding line (stimulus A1), and again rotating (around the lower point of contact between the two critical quadrilaterals) by 30° and by 45° clockwise for stimuli C1 and D1, respectively.

The patterns of row 1 were repeated three times in the successive lines with the following variations:

- a horizontal segment which coincided with the top sides of the two quadrilaterals (row 2);

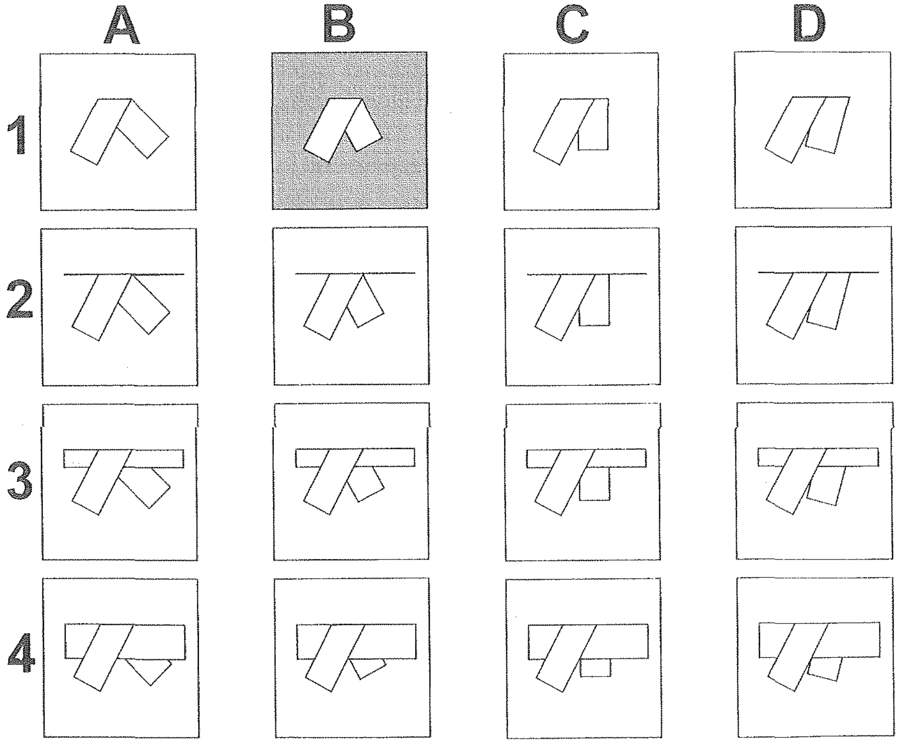


Fig. 8. Stimuli used in the second experiment.

- a horizontal rectangle that measured 1.5 cm in height (row 3);
- a horizontal rectangle that measured 2.7 cm in height (row 4).

The four level variable that differentiates the columns of Fig. 8 was named slope, while the four level variable that differentiates the rows of Fig. 8 was called interposed element.

### *3.1.3. Procedure*

The 16 stimuli were shown to the subjects in groups of four at a time, so that during each trial the subjects would see either four stimuli having the same slope, but a different interposed element (columns in Fig. 8) or four stimuli having the same interposed element, but different slopes (rows in Fig. 8). Each stimulus was thus viewed twice by each subject. The presentation order of the quadruplets was randomized and the presentation of column stimuli and row stimuli was alternated. A quadruplet was placed on a table in random order and the subjects' task was to order them according to the impression of belongingness of the two critical elements to the same object. Once the ranking was done, the participants were asked to evaluate the intensity of the cohesion between the two critical elements, using a 10 step scale, as in the previous experiment. Each subject performed a training session with different patterns before the actual experimental session began.

### 3.2. Results

The rankings made by each subject were analyzed using Kendall's $W$ agreement test. Table 3 shows the results concerning the presentation of quadruplets, where the slope was held constant (columns in Fig. 8), while Table 4 shows the results related to quadruplets, where the interposed element was held constant (rows in Fig. 8).

If subjects showed a high agreement in their rankings, then they must have used the same ordering criterion. Table 3 shows that this was the case for the patterns belonging to columns C and D in Fig. 8, but not for those belonging to columns A and B. In the latter cases there was no common ordering criterion, since there was no difference among the stimuli of each column, as far as the cohesion of the two quadrilaterals was concerned.

Table 4 shows the agreement level (Kendall's $W$ test) reached by subjects in their ordering of the four patterns of each presentation when they presented the same interposed element (rows 1, 2, 3, 4 in Fig. 8).

This table shows that subjects used the same ordering criterion for each condition related to the interposed element. In summary then, it can be concluded that, when the patterns to order have the same interposed element, then the ordering criterion is based on the degree of slope between the two quadrilaterals.

The evaluations of the degree of cohesion of the two quadrilaterals, expressed by using a 10 step scale, were analyzed by means of an ANOVA for repeated measures. The variables considered were: (1) the degree of slope of the right quadrilateral

Table 3
The agreement level (Kendall's $W$ agreement test) reached by subjects in ordering the four patterns having the same slope

| Slope | Stimuli | $W$ | Chi-squared | $S$ |
|---|---|---|---|---|
| (–)15° | A1–A2–A3–A4 | 0.013 | 0.62 | NS |
| 0° | B1–B2–B3–B4 | 0.016 | 0.77 | NS |
| 30° | C1–C2–C3–C4 | 0.44 | 21.12 | <0.001 |
| 45° | D1–D2–D3–D4 | 0.98 | 47.04 | <0.001 |

Table 4
The agreement level (Kendall's $W$ agreement test) reached by subjects in ordering the four patterns having the same interposed element

| Element | Stimuli | $W$ | Chi-squared | $S$ |
|---|---|---|---|---|
| None | 1A–1B–1C–1D | 0.72 | 34.56 | <0.001 |
| Line | 2A–2B–2C–2D | 0.8 | 38.4 | <0.001 |
| Narrow rectangle | 3A–3B–3C–3D | 0.58 | 27.84 | <0.001 |
| Large rectangle | 4A–4B–4C–4D | 0.78 | 37.44 | <0.001 |

(columns A, B, C, D in Fig. 8); (2) the type of interposed element (rows 1, 2, 3, 4 in Fig. 8).

The first main effect (*slope*) was significant: $F(3,60) = 37.98$ $p < 0.0001$. This result reveals that the subjective impression of "belongingness" of the two critical elements to the same structure decreases as the clockwise slope of the right element increases from the standard condition (stimulus B1 in Fig. 8). From the post hoc tests (test of Scheffé) it emerged that the stimuli of columns A ($X = 7.33$) and B ($X = 7.93$) were evaluated as more cohesive than those of columns C ($X = 4.99$) and D ($X = 4$) and, moreover, the patterns of column C were seen to be more cohesive than those of column D. The stimuli of column B and A are typical examples of phenomenal folding (see Massironi, 1988). The stimuli C1 and C2, D1 and D2 do not have the conditions for folding and the two quadrilaterals appear as independent and superimposed.

The second main effect (*interposed figure*) was also significant: $F(3,60) = 17.1$, $p < 0.0001$.

This result shows that the degree of cohesion was statistically higher when the interposed element was a rectangle, than when it was a line or when there was not any interposed element.

No significant differences emerged between the two rectangles (rows 3, 4 in Fig 8).

## 4. Interim discussion

The first two experiments will be discussed together, using only the stimuli of the last two columns of Figs. 6 and 8, which are the crucial configurations for an analysis of completion by folding, because they present the strongest cases of the effect of interposed figure.

If we were asked to describe simply what we could see, we should talk about a rectangular lamina, from which the two ends of a rectangular band hang. This perceptual result is difficult to explain completely on the basis of our knowledge of the processes of amodal completion. The following points are discussed in order to put forward the hypothesis that amodal completion is, on the whole, a process which works at two levels, one dealing with spatial information and the other with formal aspects.

### *4.1. On the local approach*

Research by Kanizsa (1979), Kellman and Loukides (1987), Boselie (1988), Boselie and Wouterlood (1989) and Kellman and Shipley (1991) focuses on the importance of local information. In particular, Kellman and Shipley (1991) define with the term *relatability* the fact that two partially hidden patterns from an occluding form are seen to join up behind the latter, if the prolongations of their axes meet to form a very open angle (over or equal to 90°). In the cases in this work, the two critical parts are seen to constitute a single object, even if the prolongations of their sides not only do not meet under the occluding rectangle, but form angles that are much less than 90°. From the experiments clearly emerged the reciprocal position of the parts in the phenomenal space. The shape of the occluded part, which did not obey rigid rules, was not specifically defined but it suggested solutions that often vary according to circumstances. What remained undefined in our stimuli, in which there was completion even without relatability, was the shape of the folded parts and the way of their unification; there was no doubt, on the other hand, that they did indeed unify behind the occluding form.

### *4.2. On the global approach*

Best known for the role of global factors in form perception is Leeuwenberg's (1969, 1971) *Structural Information Theory* (see also Buffart, Leeuwenberg, & Restle, 1981). They have designed a regularity-based coding system to classify the degree of regularity of the different completion alternatives: the alternative with the highest degree of symmetry will coincide with the percept.

Once a method for measuring phenomenal complexity has been set up, it may be demonstrated that – in many cases, even if not in all – the perceptual result favored by observers corresponds to that which has the least perceptual complexity (Buffart et al., 1981; Hochberg & McAlister, 1953; Van Lier, 1999; Van Lier, van der Helm, & Leeuwenberg, 1995; Van Lier & Wagemans, 1999). Simplicity is revealed in the greatest regularity and symmetry of the figure which is completed.

In the case of the last two patterns of Figs. 6 and 8, the perceptual result would undoubtedly be of two rectangles, one folded on top of the other. Other information, however, such as the folding and the triple layers, are not ascribable only to the degree of the complexity of the figure. Once again the position of the three component parts in space is clear – which one is above and which below. Nevertheless, the way in which the shapes articulate in order to join up is vague and undefined. Van Lier (1999) utilizes the concept and experience of "fuzziness" to explain the completion of particularly complex occluded shapes. It may be that the perceptual mechanisms that produce fuzzy solutions are also activated in the patterns considered here, but only in relation to the shape of the partially occluded parts and not to the way they are formed at depth.

### *4.3. Folding without folding*

Our subjects were presented with a folded shape, even without the two conditions required for a phenomenal folding, i.e., the side common to the two figures that form the margin of the folding, and the convergence of the three bands at the ends of this folding (see Appendix A). When considering Fig. 6n and Fig. 8 D4, which constitute two cases of phenomenal folding, what is found to be vague is where the folding margin common to the two critical elements actually is. The upper side of the left critical element would seem to be the right choice for this, but it does not appear to be directly connected to the right quadrilateral. The result is the phenomenal experience of folding, without the necessary conditions for its existence.

### 4.4. Three-dimensional retrieval

The last four stimuli of columns D and C in Fig. 8 and the stimuli m and n of Fig. 6 constitute an example of perceived three-dimensional space of two-dimensional figures. The horizontal rectangle can, in fact, be seen as the front face of the parallelepiped on which the prolongation in depth of the left quadrilateral rests; the left quadrilateral is then joined to the right one in a second folding line. From the perceptual point of view, however, this solution is somewhat vague, because just how the unification comes about is not clear, although we do establish that it takes place. The system's degree of tolerance is remarkable when it activates the completion by folding.

In our opinion the perceptual system is not so much interested in the shape of the hidden parts of the occluded figures as much as in their reasonable "belongingness" to the same object or configuration which passes "under" an occluding figure. We believe that the shape of the occluded figures is a problem for the researchers of perception, but not for the perception. For this reason we have referred to "belongingness" of the visible parts to the same phenomenal object.

### 4.5. Going back to the perception of closed flat knots

It was stated at the beginning of this work that the patterns of Fig. 1(a) and (b) are seen as representing the outline of a knot, which means that the object under observation appears to be the result of transformations carried out on a material that was originally of a different shape. The perceptual result would, therefore, seem to be the result of two temporal phases: in the first the knot is clearly seen, in the second the initial shape and the subsequent transformations are perceived in a fuzzy way. We are aware, that is, that it occurs as a result of transformations, but we are unable to say either how many there are or what they are like.

In order to discover the factors that promote this perceptual result, the knot was undone and a particular type of completion was seen to occur along the three folding lines: completion by folding. Research on the knots was momentarily suspended in order to establish, through two experiments, how this perceptual phenomenon works.

What had to be established was whether, and how, completion by folding contributed to the perception of CFKs. For this purpose the problem of the relation between their geometrical and perceptual result was reviewed.

The points to be considered may be summarized as follows:

1. Do other CFKs exist besides the prototype considered here?
2. If so, what are their characteristic geometrical conditions?
3. Are geometrically correct knots always perceived as such?
4. If not, what are the necessary factors for the perception of the knot?

In order to answer these questions an empirical approach was first required, i.e., to create a large number of representations of knots which introduce systematic variations on the prototype knot, and then proceed to their classification. The first element to which the variations were applied was the angle at the vertex of the knot. We then changed the reciprocal position of the three folding lines, followed by changes in their length. The figures thus obtained then underwent rotation of the prolongations around the three folding lines until a two-dimensional unfolded shape was obtained. This experiment was presented by Massironi and Fregonese (1998) and is here summarized in Appendix B.

## 5. Experiment 3

The aim of this experiment was to identify the conditions that are necessary so that a given pattern would be seen as representing a CFK. A further aim was to try and understand the relationship that exists between visibility and feasibility of a CFK.

### *5.1. Methods*

#### *5.1.1. Subjects*

Twenty students from the University of Verona, aged between 18 and 29 (10 females and 10 males), took part in the experiment.

#### *5.1.2. Material*

Two variables were considered: (1) geometrical feasibility of the knot (two levels feasible–not feasible); (2) probability to be recognized as a knot (two levels: high probability–low probability).

The two by two combinations of these variables gave rise to the four following possibilities:

1. Representation of geometrically feasible knots, with a high probability of being perceived as such.
2. Representation of geometrically feasible knots, with a low probability of being perceived as such.
3. Patterns which have a high probability of being perceived as knots, but which are not feasible from a physically geometrical point of view.
4. Pseudo-knots or rather patterns similar to knots that are not geometrically feasible, and which have a low probability of being perceived as such.

Despite the accuracy of the appropriate stimuli, the patterns which corresponded to point number 3 could not be created.

This difficulty led us to believe that it is not possible to create patterns of geometrically unfeasible knots, which in spite of this, are still recognized as knots.

The material used in the experiment was composed of 21 white cardboard cards that measured $21 \times 21$ cm$^2$ on which were outlined, in ink, the three following groups of figures:

(a) Seven drawings of knots that were actually feasible and which we believed to be recognizable [Fig. 9(A) a–g]. They are knots obtained from the prototype

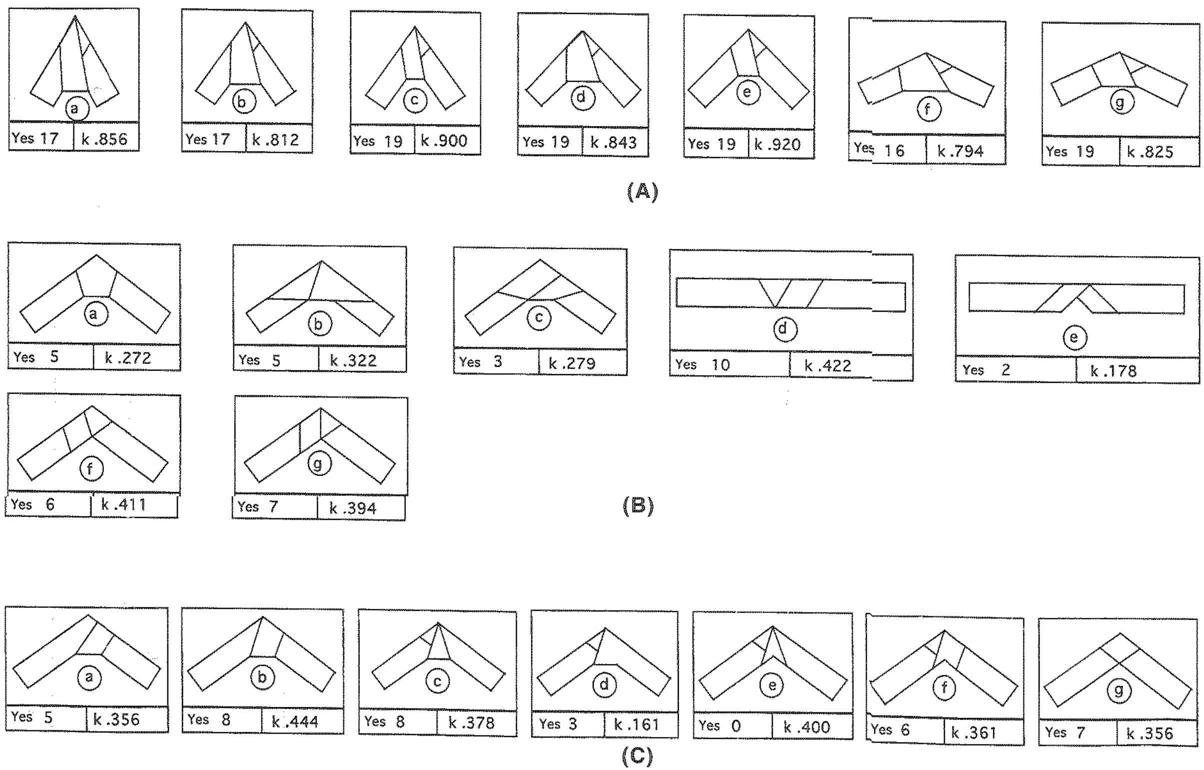


Fig. 9. Material and results of experiment 3. Drawings of geometrically correct and perceptually evident knots (A); drawings of geometrically feasible, but not perceptually evident, knots (B); drawings of pseudo-knots (C). Explanation in the test.

of Fig. 1(b) by changing the width of the angle at the vertex and the length of folding line 1 and by maintaining the alignments among the points. The angle at the vertex is 60° for knot 'a'; 72° for the pair 'b' and 'c'; 90° for the pair 'd' and 'e'; 130° for the pair 'f' and 'g'.

(b) Seven drawings of knots that were actually feasible, which did not, however, respect all the alignments and connections of Fig. 1(d) and which were believed to be unrecognizable [Fig. 9(B) a–g].

(c) Seven drawings similar in many aspects to CFKs, but which depicted knots that were not feasible since they did not respect all the constraints that characterize all CFKs (see Appendix B) [Fig. 9(C) a–g].

It was hypothesized that only stimuli belonging to the first group would be seen as representations of knots by most of the subjects.

*5.1.3. Procedure*

During the training session, each participant was shown four images of knots, three of which were geometrically feasible. An explanation was then given as to the meaning of a closed flat knot and the subjects were shown three paper models of knots that they had previously seen as drawings. These models were "unfolded" and "re-knotted" in front of their eyes. Finally the participants were told that the task consisted in deciding whether the drawings they were going to see represented a knot or not. After each judgment the participant was asked to give a subjective assessment of the degree of certainty of each reply on a scale ranging from 1 to 5, in which 1 indicated "very unsure" and 5 indicated "very sure". The 21 cardboard cards represented in Fig. 9(A)–(C), mixed altogether, were presented, one at a time, in a completely random fashion to all subjects.

*5.2. Results*

Fig. 9(A)–(C) shows, beneath each stimulus, the frequencies with which this stimulus was recognized (Yes) as a knot. All the patterns in Fig. 9(A) had a higher number of positive replies than the negative ones, whereas all the patterns of Fig. 9(B) and (C) had a decidedly higher number of negative replies. This result indicates that when at least two of the alignment/connections previously considered are not present, the knots, even if feasible, are not recognized as such. This means that, even if from a physico-geometrical point of view, the knots can be distinguished from non-knots, the same distinction cannot be made at a perceptual level. Knots, which are perceived as such, are always feasible, but the contrary is not true, i.e., not all feasible knots are also perceived knots.

It should be remembered that the stimuli in Fig. 9(A) are the only ones that show the alignments among the points [see Fig. 1(d)] and they are still the only ones that show completion by folding corresponding to the three FLs.

The possibility that each pattern had of being perceived as a knot was calculated by using the evaluation of certainty that the subjects had attributed to their answers.

To every value, expressed by means of a five stage scale, a constant of 0.5 was subtracted, and the sign + (plus) or the sign − (minus) was attributed depending on whether it was assigned to the recognition or non-recognition of a knot.

If, for example, a pattern had received an assessment of 3 regarding the degree of certainty of recognition of the knot, it was transcribed as +2.5; if associated to a reply of non-recognition of the knot, it was transcribed as −2.5. A constant of 4.5 was added to all the scores transformed in this way so that a measure between 0 and 9 was obtained. To avoid those subjects who had used wider ranges of evaluation from having more weight than those who had avoided extreme scores, the data obtained were further transformed on the basis of the following formula: $k = (K - K_{\min})/(K_{\max} - K_{\min})$. Every "$k$" value is the *knotness rating* of the stimulus it refers to, and is reported beneath each stimulus on the right of Yes frequencies in Fig. 9(A)–(C).

A non-parametric analysis of these knotness ratings confirmed the analyses from ANOVA on the log frequencies.

### *5.3. Discussion*

The results of this experiment reveal that the group of all the geometrically feasible knots includes a subgroup of knots which are perceived as such. They present the following characteristics: (a) they show the alignments and the correspondence between the points, as indicated in Fig. 1(d); (b) they present the conditions of completion by folding corresponding to the three-folding lines; (c) FL2 must be equal to FL3 and have a common point of origin; (d) FL1 must be perpendicular to the bisector of the angle at the vertex of the knot.

On the one hand, what is striking when dealing with phenomenal knots is the great number of bonds, correspondences and geometrical relationships that characterize them; on the other hand, the difficulty in understanding whether, and how, the perceptual activity takes these bonds and relationships into account constitutes one of the problems which is peculiar to this phenomenon.

A knot will be seen as such, only if the margins of the figure considered do not all appear to be the same. In fact, the FLs appear as different salient parts of the contour, since the surfaces they delimit are seen to continue behind themselves. Although the patterns of Fig. 9(b) all represent geometrically feasible knots, they do not appear as knots, because the FLs are not clearly discernible from the other lines.

The third experiment allowed us to verify which factors are necessary so that the outline drawings are perceived as representing CFKs. We are, therefore, able to say whether the drawing will be perceived as a knot, but we are unable to explain why this is so.

## 6. Conclusions

Before this study only one type of CFK was known, the one shown in Fig. 1(a) and (b). Other types of knots were known and used, like the complex, abstract ones of topology (Sossinsky, 1999). There also exists a substantial catalog of knots invented and used by sailors and mountain climbers, as well as a series of empirical solutions, just as precious, used for conjunctions, clamps, wrappings, linings, coverings, and decoration (clothes, buttonholes, slip-knots/nooses, etc.). There is, moreover, a whole set of knots at the service of the fashion industry like those for ties and belts (Fink & Mao, 1999).

The closed flat knots studied constitute a remarkable perceptual problem. We see them as the result of a transformation of something that was originally of a different shape. The relationship between closed knots and their unfolded shape is still an open question in the study of the perception of knots and constitutes a subject for further specific research.

The apparent lack of theoretical tools at our disposal makes it difficult to find a solution to the problem raised in this research. In cases like this it is normal procedure for scientific research to place new problems in a theoretical setting which is open to discussion. Thus, the problem of phenomenal knots seems related to the study of amodal completion.

In some way our results seem to be in agreement with the hypothesis put forward by Van Lier (1999). He showed that the completion of partially occluded figures, with particularly complex contours, would be seen as imprecise and fuzzy and the observer would not be able to establish precisely what happened to the margins, which are the objects of completion. Van Lier considered the hidden contour must be there but just exactly what it is like, can only be guessed. In our case, the existence of an original figure (unfolded knot) is intuitively recognized but its characteristics can only be guessed at in a fuzzy and imprecise way.

It is tempting, however, to say that a sort of fuzziness occurs in the phenomena of amodal completion, whenever particularly complex stimuli and elaborations are involved. In the typical cases of amodal completion, we see one shape in front of another and both are perceived as whole and independent. The knots, instead, do not appear to be made up of forms that were first unified and then layered. It would rather seem to be a process that follows the opposite direction, a process in which the layering comes before the unification: the final result is that of a single object composed of variously layered parts, and not layers of variously joined forms. In the case of perception of knots, the "layering" does not take place only in space, but also in time: the observed shape seems to be the result of a transformation. In order to see a transformation in a static shape, at least two moments of the observed object must be perceived: the system must be able to retrieve, from the figure being observed, some aspects of the shape, before the transformation process. There is a sort of temporal layering, since what we are observing conceals in its present shape – and at the same time reveals – aspects of the initial shape, which, as has been seen, is not retrievable.

In Fig. 10 are reported the open shapes of four knots, "b", "c", "e", "f" of Fig. 9(A). The knots are made of these open shapes, but it is almost impossible to perceive them in the knots themselves. In these cases the approximate and imprecise shapes assume the fuzzy characteristic that Van Lier talks about. It is, however, a gradual, not absolute, type of uncertainty, in that it is most likely influenced by the presence or absence of characteristics, like regularity, symmetry and parallelism,

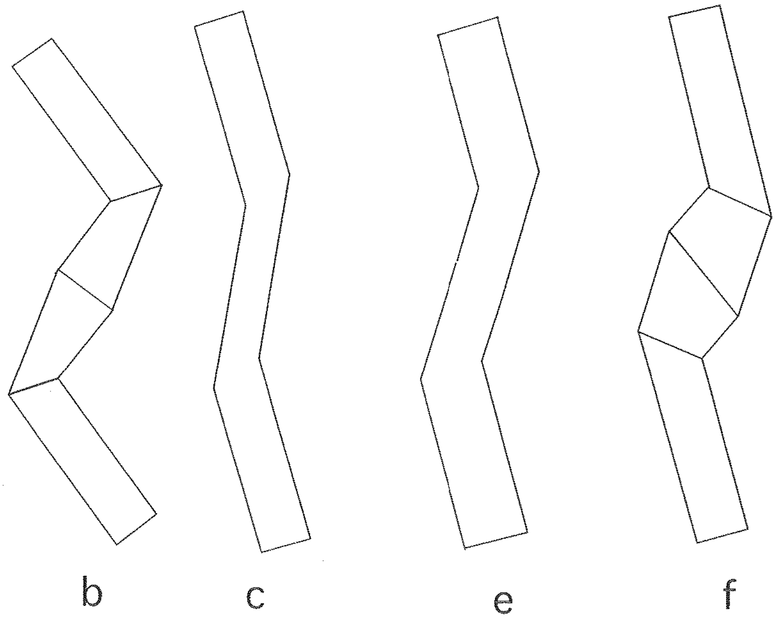


Fig. 10. Unfolded forms of patterns "b", "c", "e", "f", of Fig. 9(A).

that contribute to a greater simplification of visual stimuli (see also Wagemans, 1992, 1993). In fact, if the original shapes of the knots "b" "e" "c", having the same angle at the vertex of the knot of 72° are considered, it may be observed that the knotness rating is higher for the simplest one, i.e., the one whose sides of the central band are parallel. In the other pairs of figures that also have the same angle at the vertex, i.e., ("d"–"e" = 90°) and ("f"–"g" = 130°), the simplest configuration achieves the highest knotness rating. It is noteworthy that the most regular knots also have a more regular unfolded shape.

The aim of this work was to try and solve some of the closely related problems involved in the perception of CFKs. The purpose was not to put forward complex hypotheses on the mechanisms that regulate the integration and coordination of the phenomenon, but to run along the less complicated path of comparison between the geometrical aspects of the configurations used and the phenomenal evidence of the perceptual result.

The first aspect considered was amodal completion by folding, without which the problem of closed flat knots would be difficult to approach: CFKs would not occur without completion by folding corresponding to three folding lines. This notwithstanding, it cannot be said that this piece of information is actually used by the perceptual system when the perceptual result is a phenomenal knot.

A second aspect involves the specific information of the folding lines that appear as graphic signs with particular characteristics which represent the side common to the two strips of a singlefolded shape placed, however, on two different planes. When the FLs are not distinguishable from the other outlines of the configuration, the CFKs are not recognizable, even if they are geometrically feasible knots.

The mechanism involved here may be one that carries out an edge analyzing system, and is able to distinguish the different meaning one line may have in relation to its context, (Gibson, 1979; Kennedy, 1974). A mechanism of this kind may work in the same way, as the one suggested by Cutting and Massironi (1998), concerning graphic outlines.

A third aspect deals with the alignments and correspondences interconnected between the noticeable points of the CFKs. In this case too, there may be a characteristic that the observer is unaware of, which is, however, present in all CFKs seen as such, and absent in feasible knots not visually recognizable.

The last aspect involves the phenomenal presence of a temporal dimension which accompanies the perception of CFKs. It is a fuzzy process since we are not completely aware of the temporal components, which actually characterize the knot.

**Acknowledgements**

We wish to thank the section editor, Johan Wegemans, and the two referees, Michael Kubovy and Rob Van Lier, for their insightful observations and remarks.

This research received financial support from the Italian MURST (Ministry of University and Scientific and technological Research).

**Appendix A**

The structural elements that produce the phenomenal result of folding are the following four:

1. The existence of two phenomenally overlapping figure areas.
2. The two consecutive vertexes belonging to the perimeter of such figures and segment lying between them must coincide. This common segment is the folding line (FL).

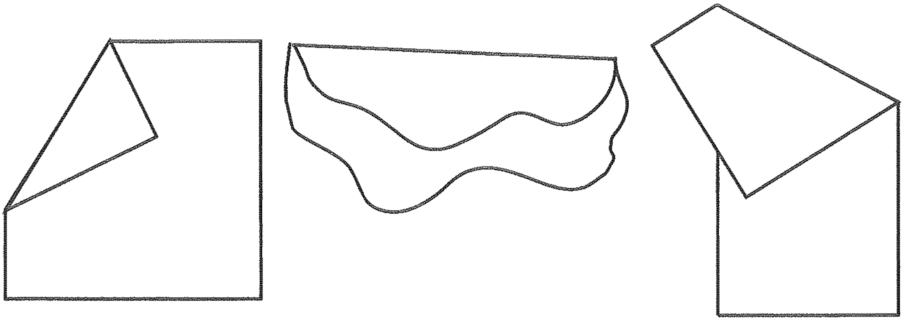

Fig. 11. Three cases of phenomenal folding.

3. The two overlapping figures must be on the same side of the folding line.
4. Three segments must converge at the extreme points of the folding line in an arrow-like configuration in either of the two following ways: (a) all three segments are visible or (b) only two of them are visible and a part of the third segment can be amodally seen. This part (if extended) would reach the point of convergence of the other two segments.

See patterns of Fig. 11.

**Appendix B**

The mandatory elements that must be respected in order to represent a geometrically feasible closed flat knot concern the reciprocal relations between the three folding lines (FLs). The necessary conditions to be able to draw a closed flat knot are as follows:

1. The presence of three segments (the three folding lines) laid out on a plane in such a way that both ends of one of the segments may be connected to the homologous

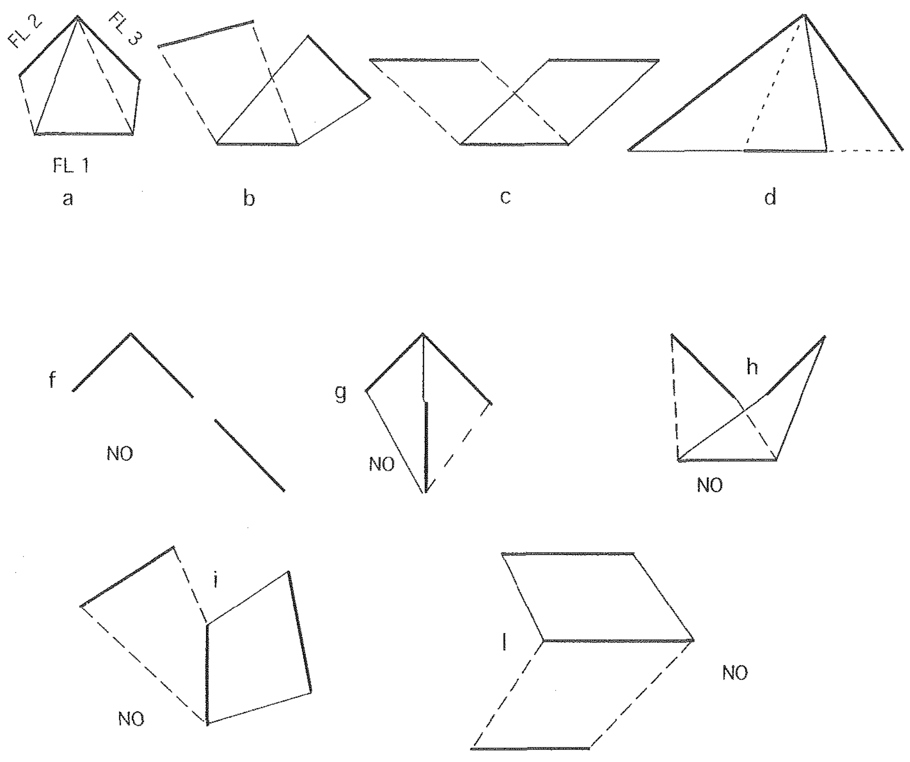


Fig. 12. Reciprocal position of the three geometrically correct (a, b, c, d), and geometrically incompatible FLs (f, g, h, i, l). Explanation in the text.

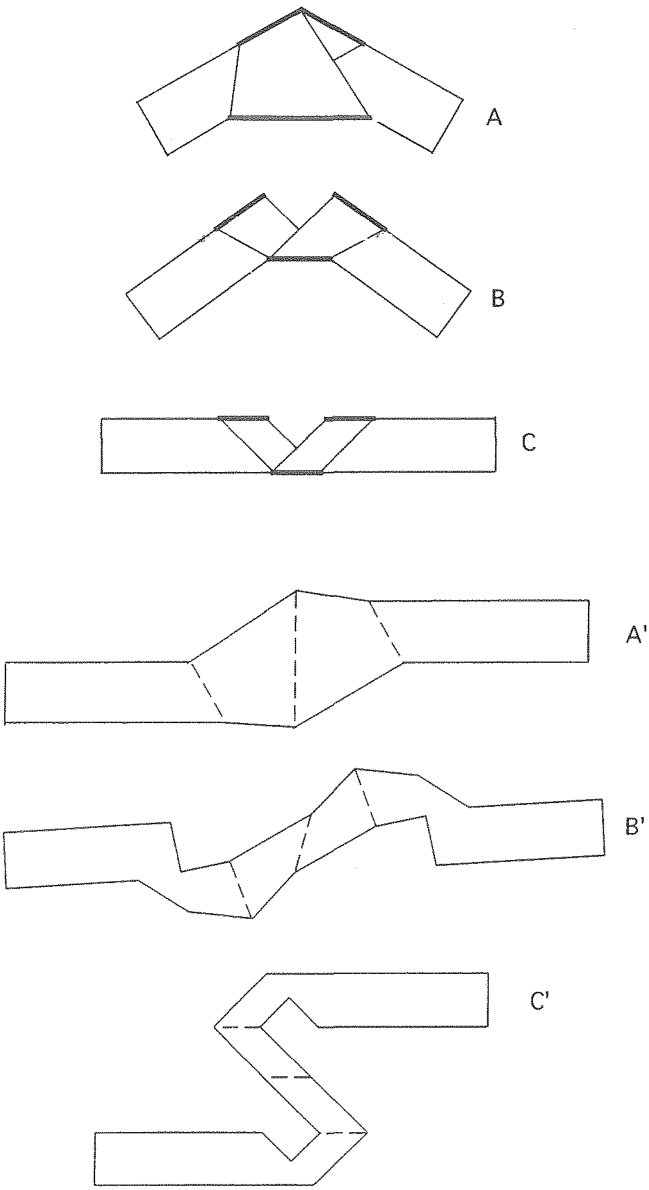


Fig. 13. Three examples of geometrically correct knots (a, b, c, d), and the strips from where they originate.

ends of the other two segments as in Fig. 12a, b, c, d but not as in Fig. 12f, h. The segment connected to the other two is FL1 (Fig. 12a).

2. The three FLs must be placed in such a way that should they be prolonged they form a triangle as in Fig. 12a, b, d, but not as in Fig. 12f, g, h; or they must be parallel (Fig. 12c).
3. The connections between the ends of FL1 and those of the other two FLs must be on the same side compared to FL1 as may be seen in Fig. 12a–d, but not in Fig. 12f, h, i.
4. The outward extensions of FL2 and FL3 usually, but not always, constitute the upper side of the arms, whereas the lower sides are made up of the parallels to them, leading from the ends of FL1. This point is not mandatory. Fig. 13 shows four examples of geometrically feasible knots (a, b, c, d), and the strips from where they originate.

## References


Adams, C. C. (1994). *The knot book*. New York: Freeman.

Agresti, A. (1996). *An introduction to categorical data analysis*. New York, NY: Wiley.

Boselie, F. (1988). Local versus global minima in visual pattern completion. *Perception and Psychophysics, 43*, 431–445.

Boselie, F., & Wouterlood, D. (1989). The minimum principle and visual pattern completion. *Psychological Research, 51*, 93–101.

Buffart, H., Leeuwenberg, E., & Restle, F. (1981). Coding theory of visual pattern completion. *Journal of Experimental Psychology: Human Perception and Performance, 7*, 241–274.

Cutting, J. E., & Massironi, M. (1998). Pictures and their special status in perceptual and cognitive inquiry. In J. Hochberg (Ed.), *Perception and cognition at century's end* (pp. 137–168). New York: Academic Press.

Fink, T., & Mao, Y. (1999). *The 85 ways to tie a tie*. London: Fourth Estate Limited.

Gibson, J. J. (1966). *The senses considered as a perceptual system*. Boston, MA: Houghton Maffin.

Gibson, J. J. (1979). *The ecological approach to visual perception*. Boston, MA: Houghton Maffin.

Hochberg, J. E., & McAlister, E. (1953). A quantitative approach to figural goodness. *Journal of Experimental Psychology, 46*, 361–364.

Kanizsa, G. (1979). *Organization in vision*. New York: Praeger.

Kellman, P. J., & Loukides, M. G. (1987). An object perception approach to static and kinetic subjective contours. In S. Petry, & G. E. Meyer (Eds.), *Perception of illusory contours*. New York: Springer.

Kellman, P. J., & Shipley, T. F. (1991). A theory of visual interpolation in object perception. *Cognitive Psychology, 23*, 141–221.

Kennedy, J. M. (1974). *A psychology of picture perception – images and information*. Jossey-Bass Inc.

Leeuwenberg, E. (1969). Quantitative specification of information in sequential patterns. *Psychological Review, 76*, 216–220.

Leeuwenberg, E. (1971). A perceptual coding language for visual and auditory patterns. *American Journal of Psychology, 84*, 307–349.

Lowe, D. G. (1987). Three dimensional object recognition from single two-dimensional images. *Artificial Intelligence, 31*, 355–395.

Massironi, M. (1988). A new visual problem: phenomenic folding. *Perception, 17*, 681–694.

Massironi, M., & Bruno, N. (1997). The perception of surface folding in static and animated displays. *Perception, 26*, 153–170.

Massironi, M., & Fregonese, A. (1998). On seeing knots. In M. Zanforlin, & L. Tommasi (Eds.), *Research in perception* (pp. 109–126). Saonara (PD): Logos Editore.

Michotte, A., Thinès, G., & Crabbé, G. (1964). *Les compléments amodaux des structures perceptives. Studia psychologica*. Louvain: Publications Universitaires Louvain.

Perkins, D. N. (1972). Visual discrimination between rectangular and nonrectangular parallelopipedes. *Perception and Psychophysics, 12*, 396–400.

Perkins, D. N., & Cooper, R. (1980). How the eye makes up what the light leaves out. In M. Hagen (Ed.), *The perception of pictures: vol. II. Durer's device: beyond the projective model of pictures*. New York: Academic Press.

Proffitt, D. R., Rock, I., Hecht, H., & Schubert, J. (1992). The stereokinetic effect and its relations to the kinetic depth effect. *Journal of Experimental Psychology: Human Perception and Performance, 18*, 3–21.

Ramachandran, V. S. (1985). Guest editorial. *Perception, 14*, 97–103.

Rock, I. (1983). *The logic of perception*. Cambridge, MA, London: The MIT Press.

Sekuler, A. B., & Palmer, S. E. (1992). Perception of partly occluded objects: a microgenetic analysis. *Journal of Experimental Psychology: General, 121*, 95–111.

Siegel, S., & Castellan, N. J. (1988). *Nonparametric statistics for behavioral sciences*. New York: McGraw-Hill.

Sossinsky, A. (1999). *Noeuds. Genèse d'une Théorie Mathématique*. Paris: Seuil.

Uttal, W. R. (1988). *On seeing forms*. Hillsdale, NJ: Erlbaum.

Van Lier, R. (1999). Investigating global effects in visuale occlusion: from partly occluded square to the back of a tree-trunk. *Acta Psychologica, 102*, 203–220.

Van Lier, R. J., van der Helm, P. A., & Leeuwenberg, E. L. J. (1995). Competing global and local completions in visual occlusions. *Journal of Experimental Psychology: Human Perception and Performance, 21*, 571–583.

Van Lier, R., & Wagemans, J. (1999). From images to objects: global and local completions of self-occluded parts. *Journal of Experimental Psychology: Human Perception and Performance, 25*, 1721–1741.

Wagemans, J. (1992). Perceptual use of nonaccidental properties. *Canadian Journal of Psychology, 46*, 236–279.

Wagemans, J. (1993). Skewed symmetry: a nonaccidental property used to perceive visual forms. *Journal of Experimental Psychology: Human Perception and Performance, 19*, 364–380.

Wagemans, J., & Kolinsky, R. (1994). Guest editorial: perceptual organization and object recognition. POOR is the acronym, rich the notion. *Perception, 23*, 371–382.

Witkin, A. P., & Tenenbaum, J. M. (1983). On the role of structure in vision. In A. Rosenfeld, B. Hope, & J. Beck (Eds.), *Human and machine vision* (pp. 481–543). New York: Academic Press.